Inverse Problems, Design and Optimization Symposium
Rio de Janeiro, Brazil, 2004

STOCHASTIC APPROACH TO ELECTRIC PROCESS TOMOGRAPHY

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ABSTRACT

We consider the process tomography problem of imaging the concentration distribution of a given substance in a fluid moving in a pipeline based on electromagnetic measurements on the surface of the pipe. We view the problem as a state estimation problem. The concentration distribution is treated as a stochastic process satisfying a stochastic differential equation referred to as the state evolution equation. The measurements are described in terms of an observation equation containing the measurement noise. Our main interest is in the mathematical formulation of the state evolution equation. The time evolution is modelled by a stochastic convection-diffusion equation. We derive a discrete evolution equation for the concentration distribution by using the stochastic integration theory and the semigroup technique.

NOMENCLATURE

$\arg\lambda$	the argument of a complex number λ
A^*	the adjoint of an operator A
$B(E,H)$	the space of bounded linear operator from E into H
$\mathcal{B}(E)$	the Borel σ-algebra of a topological space E
C^2	twice continuously differentiable
$\bar{D}$	the closure of a set D
∂D	the boundary of a set D
$\mathcal{D}(A)$	the domain of an operator A
$H^2(D)$	the space of functions from D into $\mathbb{C}$ with square integrable weak derivatives up to order 2
$L^2(D)$	the space of square integrable functions from D into $\mathbb{C}$
$L^1(0,T;E)$	the space of Bochner integrable functions from $[0,T]$ into E
$\mathcal{L}(X)$	the distribution of a random variable X
$\mathcal{N}(m,Q)$	the Gaussian measure with mean m and covariance Q
$\mathrm{Tr}(A)$	the trace of an operator A
$UC(D)$	the space of uniformly continuous functions from D into $\mathbb{R}^n$
$UC^1(D)$	the space of uniformly countinuously differentiable functions from D into $\mathbb{R}^n$
$\mathbf{v}$	the outer unit normal vector

INTRODUCTION

We consider the process tomography problem of imaging the concentration distribution of a given substance in a fluid moving in a pipeline based on electromagnetic measurements on the surface of the pipe. In electrical impedance tomography (EIT), electric currents are applied to electrodes on the surface of an object and the resulting voltages are measured using the same electrodes (Figure 1). The conductivity distribution inside the object is reconstructed based on the voltage measurements. The relation between the conductivity and the concentration

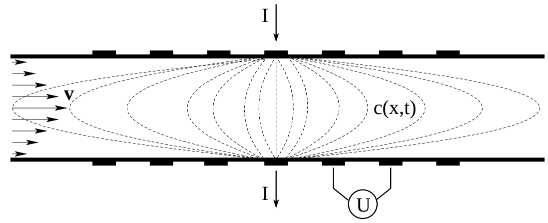

Figure 1: EIT in process tomography

depends on the process and it is usually non-linear. At least for multiphase mixtures and strong electrolytes, such relations are studied and discussed in the literature.

In traditional EIT, it is assumed that the object remains stationary during the measurement process. A complete set of measurements, also called a frame, consists of all possible linearly independent injected current patterns and the corresponding set of voltage measurements. In process tomography, in general we cannot assume that the target remains unaltered during a full set of measurements. Thus conventional reconstruction methods cannot be used. The time evolution needs to be modelled properly. We view the problem as a *state estimation problem*. The concentration distribution is treated as a stochastic process, or a state of the system, that satisfies a stochastic differential equation referred to as the *state evolution equation*. The measurements are described in terms of an *observation equation* containing the measurement noise.

Often in a state estimation approach the time variable is assumed to be discrete and the space variable to be finite dimensional. It is convenient from the practical point of view. Observations are usually done at discrete times and the compution requires space discretization. Since our interest is in the mathematical formulation of the problem, we assume that the space variable is infinite dimensional. The solution of the state estimation problem is a function valued random variable instead of an $\mathbb{R}^n$-valued Gaussian distribution.

Our goal is to have a real-time monitoring for a flow in a pipeline. For that reason the computational time has to be minimized. Therefore, we use a simple model, the convection-diffusion equation, for the flow. It is easy to implement and fast to compute. Since we cannot be sure that other features such as turbulence of the flow do not appear, we use stochastic modelling. Therefore the randomness is due to the lack of information, not to the intrinsic randomness of the concentration.

The measurements are done in a part of the boundary of the pipe. We get enough information for an accurate computation only from a segment of the pipe. It would be natural to choose the domain of the model to be the segment of the pipe. If the domain is restricted to be a segment of the pipe, we have to use some boundary conditions in the input and the output end of the segment. The choice of boundary conditions has an effect on the solution. The most commonly used boundary conditions do not represent the actual circumstances in the pipe. Therefore, we do not do the domain restriction. We assume that the pipe is infinitely long. With the assumption we derive the state evolution model. The concentration distribution, which we are actually interested in, is the restriction of the solution to the evolution model to a segment of the pipe.

This problem has been considered in the articles [1,2,3] and in the proceeding papers [4,5,6,7,8]. Those articles and proceeding papers concentrate on the numerical implementation of the problem. Our main interest is in the mathematical formulation of the state evolution equation and the estimation problem in general. We refer to those articles and references in them concerning the observation equation. The rigorous knowledge of the stochastic nature of the state evolution equation is essential for solving the electric process tomography problem.

It is understood that the estimation of distributied parameters in inverse problems may depend heavely on the discretization scheme. To develop discretization invariant estimation methods, [9], it is important to study continuous stochastic models. This is one of the main motivations of this work. Further results will be published in [10].

MATHEMATICAL PRELIMINARIES

Analytic semigroups

Let E be a Banach space. A family $\{U(t)\}$, $t \geq 0$, of bounded linear operators from E into E is called a *semigroup* if $U(t)U(s) = U(t+s)$ for all $t, s \geq 0$ and $U(0) = I$. A semigroup $U(t)$ is *analytic*, if the function $t \mapsto U(t)$ can be extended to be an analytic function from a sector

$$\left\{\lambda \in \mathbb{C} : \lambda \neq 0, \left|\arg \lambda\right| \leq \beta\right\} \tag{1}$$

with some $\beta \in (0, \pi)$ to the space of all bounded linear operators from E into E.

The linear operator A defined by

$$\mathcal{D}(A) := \left\{ x \in E : \exists \lim_{t \to 0^+} \frac{U(t)x - x}{t} \right\} \tag{2}$$

and for all $x \in \mathcal{D}(A)$

$$Ax := \lim_{t\to 0^+} \frac{U(t)x - x}{t} \tag{3}$$

is the *infinitesimal generator* of the semigroup $U(t)$.

Analytic semigroups can be used to solve initial value problems.

Theorem 1. If $U(t)$ is an analytic semigroup generated by an operator A and $u_0 \in \overline{\mathcal{D}(A)}$, then the solution to the initial value problem

$$\begin{cases} u'(t) = Au(t), \\ u(0) = u_0 \end{cases} \tag{4}$$

is

$$u(t) = U(t)u_0 \tag{5}$$

for all $t \geq 0$.

Proof. See [11] Section 5.1.2. □

Theorem 2. Let $U(t)$ be an analytic semigroup generated by an operator A with dense domain $\mathcal{D}(A)$. Let $u_0 \in E$ and $f \in L^1(0,T;E)$. Then the nonhomogeneous initial value problem

$$\begin{cases} u'(t) = Au(t) + f(t), \\ u(0) = u_0 \end{cases} \tag{6}$$

has a unique weak solution given by the formula

$$u(t) = U(t)u_0 + \int_0^t U(t-s)f(s)\, ds \tag{7}$$

for all $0 \leq t \leq T$.

Proof. See [11] Section 5.2. □

Stochastic analysis in infinite dimensions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(E, \mathcal{G})$ a measurable space. A function $X : \Omega \to E$ such that the set $\{\omega \in \Omega : X(\omega) \in A\}$ belongs to $\mathcal{F}$ for each $A \in \mathcal{G}$ is called a *random variable* from $(\Omega, \mathcal{F}, \mathbb{P})$ into $(E, \mathcal{G})$. A random variable is called *simple* if it takes only a finite number of values.

Let E be a separable Banach space. An E-valued random variable X is said to be *Bochner integrable* if

$$\int_\Omega \|X(\omega)\|\, \mathbb{P}(d\omega) < \infty. \tag{8}$$

If X is Bochner integrable, the integral $\int_\Omega X\, d\mathbb{P}$ can be defined and is denoted by $\mathbb{E}(X)$.

Proposition 3. Let E be a separable Banach space, X a Bochner integrable E-valued random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}$ a σ-algebra contained in $\mathcal{F}$. There exists a unique, up to a set of $\mathbb{P}$-probability zero, Bochner integrable E-valued random variable Z, measurable with respect to $\mathcal{G}$, such that

$$\int_A X\, d\mathbb{P} = \int_A Z\, d\mathbb{P} \tag{9}$$

for all $A \in \mathcal{G}$.

Proof. See [10] Chapter 4. □

The random variable Z is denoted by $\mathbb{E}(X|\mathcal{G})$ and called the *conditional expectation* of X given $\mathcal{G}$. We use the notation $\mathbb{E}(X|Y) := \mathbb{E}(X|\sigma(Y))$ where $\sigma(Y)$ is the σ-algebra generated by the random variable Y.

Let I be an interval of $\mathbb{R}$. A family $X = \{X(t)\}$, $t \in I$, of E-valued random variables defined on Ω is called a *stochastic process*. We set $X(t,\omega) := X(t)(\omega)$ for all $t \in I$ and $\omega \in \Omega$. Functions $X(\cdot,\omega)$ are called the *trajectories* of X.

Let $T > 0$ be fixed. A family of σ-algebras $\{\mathcal{F}_t\}$, $t \in [0,T]$, is called a *filtration*, if $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for all $s,t \in [0,T]$, $s \leq t$. We denote by $\mathcal{F}_{t^+}$ the intersection of all $\mathcal{F}_s$ where $t < s \leq T$, i.e.,

$$\mathcal{F}_{t^+} = \bigcap_{t<s\leq T} \mathcal{F}_s. \tag{10}$$

Definition 4. The filtration $\{\mathcal{F}_t\}$, $t \in [0,T]$, is said to be *normal* if
(i) $\mathcal{F}_0$ contains all $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 0$,
(ii) $\mathcal{F}_t = \mathcal{F}_{t^+}$ for all $t \in [0,T]$.

We denote by $\mathcal{P}_T$ the σ-algebra generated by sets of the form

$$\begin{cases} (s,t] \times F, \quad 0 \le s < t \le T, \ F \in \mathcal{F}_s, \\ \{0\} \times F, \quad F \in \mathcal{F}_0. \end{cases} \tag{11}$$

A measurable function from $([0,T] \times \Omega, \mathcal{P}_T)$ into $(E, \mathcal{B}(E))$ is called a *predictable process*.

Let E and H be Hilbert spaces and Q a bounded linear symmetric non-negative trace class operator.

Definition 5. A E-valued stochastic process $W(t)$, $t \in [0,T]$, is called a Q-*Wiener process* if
(i) $W(0) = 0$,
(ii) W has continuous trajectories,
(iii) W has independent increments,
(iv) $\mathcal{L}(W(t) - W(s)) = \mathcal{N}(0, (t-s)Q)$ for $0 \le s \le t \le T$.

W is a Q-Wiener process with respect to a filtration $\{\mathcal{F}_t\}$, $t \in [0,T]$, if
(i) $W(t)$ is $\mathcal{F}_t$-measurable,
(ii) $W(t+h) - W(t)$ is independent of $\mathcal{F}_t$ for all $h \ge 0$ and $t, t+h \in [0,T]$.

A $B(E,H)$-valued process $\Phi(t)$, $t \in [0,T]$, taking only a finite number of values is said to be *elementary* if there exist a sequence $0 = t_0 < t_1 < \dots < t_k = T$ and a sequence $\Phi_0, \Phi_1, \dots, \Phi_{k-1}$ of $B(E,H)$-valued simple random variables such that Φ_m is $\mathcal{F}_{t_m}$-measurable and

$$\Phi(t) = \Phi_m \quad \text{for } t \in (t_m, t_{m+1}] \tag{12}$$

for all $m = 0,1,\dots,k-1$. We define the stochastic integral for elementary processes Φ by the formula

$$\int_0^t \Phi(s)\, dW(s) := \sum_{m=0}^{k-1} \Phi_m \left(W(t_{m+1} \wedge t) - W(t_m \wedge t) \right) \tag{13}$$

where $s \wedge t = \min(s,t)$. The definition of the stochastic integral can be extended to all $B(E,H)$-valued predictable processes Φ such that

$$\mathbb{E} \int_0^T \operatorname{Tr}\left[\Phi(s) Q \Phi^*(s) \right] ds < \infty. \tag{14}$$

Let $\{\mathcal{F}_t\}$, $t \in [0,T]$, be a normal filtration and W a Q-Wiener process with respect to $\{\mathcal{F}_t\}$.

Theorem 6. Let $U(t)$ be an analytic semigroup generated by an operator A with dense domain $\mathcal{D}(A) \subset H$, f a H-valued predictable process with Bochner integrable trajectories on $[0,T]$, B a bounded linear operator from E into H and X_0 a $\mathcal{F}_0$-measurable random variable. If

$$\int_0^T \operatorname{Tr}\left[U(s) B Q B^* U^*(s) \right] ds < \infty, \tag{15}$$

then the stochastic differential equation

$$\begin{cases} dX(t) = \left[AX(t) + f(t) \right] + B dW(t), \\ X(0) = X_0 \end{cases} \tag{16}$$

has exactly one weak solution given by the formula

$$X(t) = U(t) X_0 + \int_0^t U(t-s) f(s)\, ds + \int_0^t U(t-s) B\, dW(s) \tag{17}$$

for all $0 \le t \le T$.

Proof. See [10] Chapter 4. □

STATE ESTIMATION

Let $D \subset \mathbb{R}^n$ be a domain that corresponds to the object of interest. We denote by $X = X(t,x)$, $x \in D$, a distributed parameter describing the state of the object – the unknown distribution of a physical target – at time $t \ge 0$. We assume that we have a model for the time evolution of the parameter X. We assume that instead of a deterministic function X is a stochastic process satisfying a stochastic differential equation. This allows us to incorporate phenomena such as modelling uncertainties into the model.

Let $Y = Y(t)$ denote a quantity that is directly observable at times $t \in I$, $I = \{t_k : t_k < t_{k+1}, k \in \mathbb{N}\}$. We assume that the dependence of Y upon the state X is known up to observation noise and modelling errors.

The state estimation system consists of a pair of equations

$$dX(t) = F(t,X,R) + dW(t), \tag{18}$$

$$Y(t) = G(t,X,S), \quad t \in I. \tag{19}$$

Equation (18) is called the *state evolution equation* and is to be interpreted as a stochastic differential equation in which the function F is the evolution model function and $R = R(t)$ and $W = W(t)$ are stochastic processes. The process W is called the *state noise*. Equation (19) is called the *observation equation*. The function G is the observation model function and $S = S(t)$ is a stochastic process, the *observation noise*.

The state estimation problem can be formulated as follows: *Estimate the state* X *satisfying an evolution equation of the type* (18) *based on the observed values of* $Y(t)$, *when* t *is in a given subset of* I. The most commonly used estimators are the predictor

$$\mathbb{E}\left(X(t_k) \middle| Y(t_i), i = 1,\dots,k-1\right), \tag{20}$$

the filter

$$\mathbb{E}\left(X(t_k) \middle| Y(t_i), i = 1,\dots,k\right) \tag{21}$$

and the smoother

$$\mathbb{E}\left(X(t_k) \middle| Y(t_i), i \in I\right). \tag{22}$$

Predictor (20) is based on the history at the previous time step, Filter (21) on the current history and Smoother (22) on the whole measurement set.

We consider the special case in which observations are obtained by EIT measurements and in which the physical target can be described by the convection-diffusion equation.

MATHEMATICAL FORMULATION OF THE STATE EVOLUTION EQUATION

We examine a concentration distribution in a fluid moving in a pipe with a velocity distribution defined by the laminar flow equation by doing electric measurements at the boundary of the pipe. Let $\kappa = \kappa(x)$ be the diffusion coefficient and $\mathbf{v} = \mathbf{v}(x)$ be the velocity of the flow. The diffusion coefficient and the velocity distribution are assumed to be known and stationary. In addition, the incompressibility condition $\nabla \cdot \mathbf{v} = 0$ is valid in D and the flow is tangential at the pipe walls, i.e., $\mathbf{v} \cdot \mathbf{v} = 0$ at ∂D. We assume that the concentration distribution $C(t)$ is a stochastic process satisfying the stochastic differential equation

$$dC(t) = \left[LC(t) + f(t)\right] + BdW(t) \tag{23}$$

for every $t > 0$ with the initial value

$$C(0) = C_0. \tag{24}$$

The operator L is the deterministic convection-diffusion operator

$$\begin{aligned} L : \mathcal{D}(L) &\to L^2(D) \\ c &\mapsto \nabla \cdot (\kappa \nabla c) - \mathbf{v} \cdot \nabla c \end{aligned} \tag{25}$$

with the domain

$$\mathcal{D}(L) = \left\{ c \in H^2(D) : \left.\frac{\partial c}{\partial \nu}\right|_{\partial D} = 0 \right\} \tag{26}$$

where $D \subset \mathbb{R}^n$ is an infinitely long pipe. The boundary condition at the boundary of the pipe is included in the domain of the operator L. We assume that there is no diffusion through the pipe walls. We model with $f(t)$ a possible control of the system. Since the control term is known, if the state of the system is known, we may assume that $f(t)$ is an $L^2(D)$-valued predictable process. The term $BdW(t)$ is a source term representing possible modelling errors, where B is a bounded linear operator from $L^2(D)$ to itself and $W(t)$ is an $L^2(D)$-valued Wiener process.

We use the semigroup technique to solve the stochastic convection-diffusion equation (23) with the initial value (24).

Theorem 7. Let the domain $D \subset \mathbb{R}^n$ be open and its boundary C^2 smooth. Then the operator L generates an analytic semigroup $U(t)$, if the diffusion coefficient κ is positive and bounded from below, $\kappa(x) \geq \delta > 0$ for all $x \in \bar{D}$, and the diffusion coefficient and the velocity of the flow satisfy the conditions

$$\begin{cases} \kappa : \bar{D} \to \mathbb{R}, & \kappa \in UC^1(\bar{D}), \\ \mathbf{v} : \bar{D} \to \mathbb{R}^n, & \mathbf{v} \in UC(\bar{D}). \end{cases} \tag{27}$$

Proof. See [10] Chapter 5. □

Consequently, if $c_0 \in L^2(D)$ and $f \in L^1(0,T;L^2(D))$, by Theorem 2 the deterministic version

$$\begin{cases} c'(t) = Lc(t) + f(t), \\ c(0) = c_0 \end{cases} \tag{28}$$

of the stochastic convection-diffusion equation (23) with the initial value (24) has a unique weak solution

$$c(t) = U(t)c_0 + \int_0^t U(t-s) f(s)\, ds \tag{29}$$

for all $t \in [0,T]$.

By Theorem 6 the weak solution to the stochastic convection-diffusion equation (23) with the initial value (24) has the similar form as the deterministic one.

Theorem 8. Let $\{\mathcal{F}_t\}$, $t \in [0,T]$, be a normal filtration and W a Wiener process with respect to the filtration. If f has Bochner integrable trajectories on $[0,T]$ and C_0 is $\mathcal{F}_0$-measurable, then the stochastic convection-diffusion equation (23) with the initial value (24) has exactly one weak solution given by the formula

$$\begin{aligned} C(t) = U(t)C_0 + \int_0^t U(t-s) f(s)\, ds + \\ + \int_0^t U(t-s) B\, dW(s) \end{aligned} \tag{30}$$

for all $t \in [0,T]$.

Proof. See [10] Chapter 5. □

Discrete evolution model without control

We assume that there is no control in our system, i.e., $f \equiv 0$. The measurements are done in a discrete set of times t_k. We use the notation $C_k = C(t_k)$ and $\Delta_k = t_{k+1} - t_k$. Then a discrete evolution model for the concentration distribution is

$$C_{k+1} = U(\Delta_k) C_k + W_k \tag{31}$$

where

$$W_k = \int_{t_k}^{t_{k+1}} U(t_{k+1} - s) B\, dW(s). \tag{32}$$

The term W_k can be seen as a state noise.

The discrete evolution model (31) combined with an observation model enables us to calculate some estimator for the concentration distribution.

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